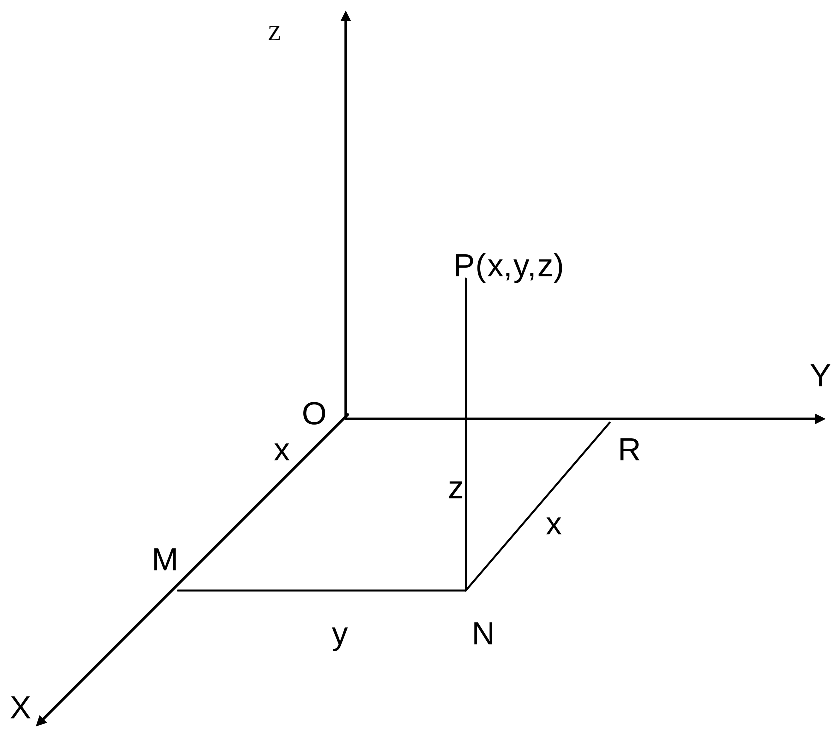
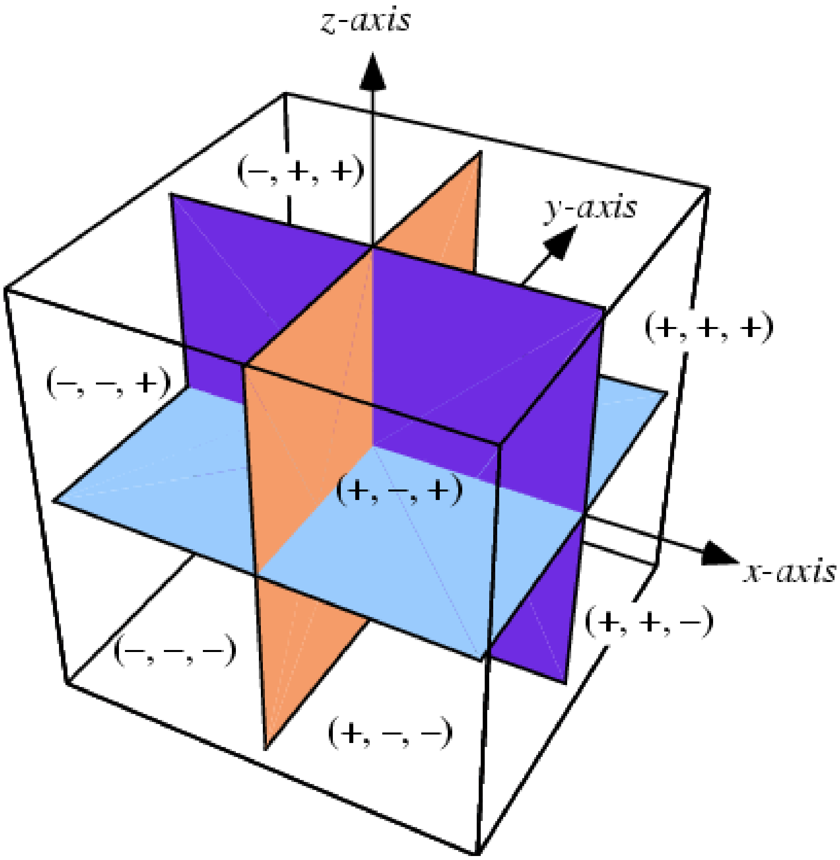
Three dimensional Analytical geometry

Let OX ,OY & OZ be mutually perpendicular straight lines meeting at a point O. The extension of these lines OX^1, OY^1 and OZ^1 divide the space at O into octants(eight). Here mutually perpendicular lines are called X, Y and Z co-ordinates axes and O is the origin. The point P (x, y, z) lies in space where x, y and z are called x, y and z coordinates respectively.



where NR = x coordinate, MN = y coordinate and PN = z coordinate



Distance between two points

The distance between two points $A(x_{1,} y_{1,} z_1)$ and $B(x_{2,} y_{2,} z_2)$ is

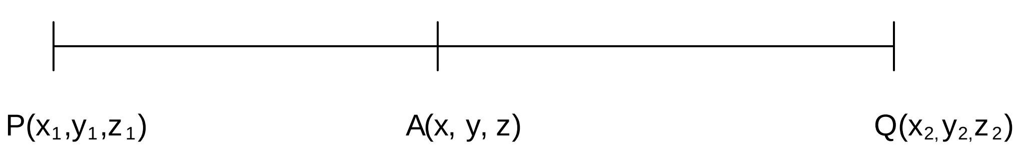
dist AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

In particular the distance between the origin O (0,0,0) and a point P(x,y,z) is

OP = $\sqrt{x^2 + y^2 + z^2}$

The internal and External section

Suppose $P(x_{1,} y_{1,} z_1)$ and $Q(x_{2,} y_{2,} z_2)$ are two points in three dimensions.

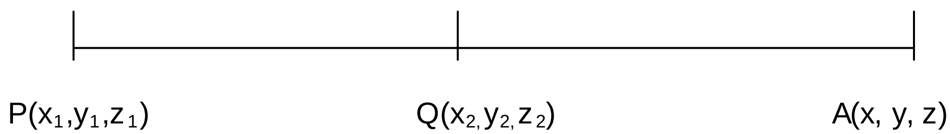


The point A(x, y, z) that divides distance PQ internally in the ratio m_1:m_2 is given by

$$A = \left[\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right]$$

Similarly

$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in three dimensions.



The point A(x, y, z) that divides distance PQ externally in the ratio $m_1:m_2$ is given by

$$A = \left[\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}, \frac{m_1z_2-m_2z_1}{m_1-m_2}\right]$$

If A(x, y, z) is the midpoint then the ratio is 1:1

$$A = \left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right]$$

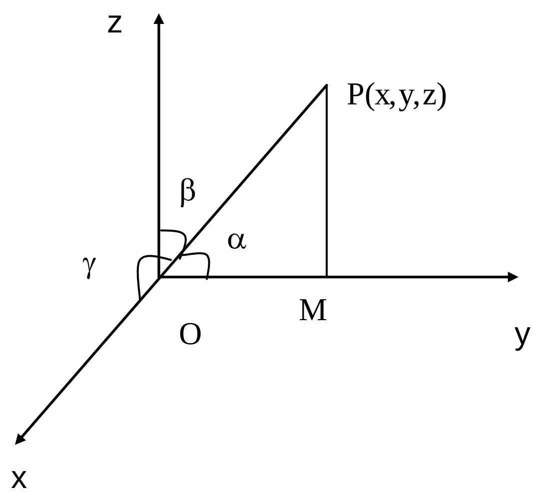
Problem

Find the distance between the points P(1,2-1) & Q(3,2,1)

PQ= $\sqrt{(3-1)^2+(2-2)^2+(1+1)^2} = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

Direction Cosines

Let P(x, y, z) be any point and OP = r. Let α,β,γ be the angle made by line OP with OX, OY & OZ. Then α,β,γ are called the direction angles of the line OP. cos α, cos β, cos γ are called the direction cosines (or dc's) of the line OP and are denoted by the symbols l, m ,n.



Result

By projecting OP on OY, PM is perpendicular to y axis and the $\angle POM = \beta$ also OM = y

$$\therefore \cos\beta = \frac{y}{r}$$

Similarly, $$\cos\alpha = \frac{x}{r}$$

$$\cos\gamma = \frac{z}{r}$$

(i.e) $l = \frac{x}{r}$, $m = \frac{y}{r}$, $n = \frac{z}{r}$

$$\therefore l^2 + m^2 + n^2 = \frac{x^2 + y^2 + z^2}{r^2}$$

($\because r = \sqrt{x^2 + y^2 + z^2} \Rightarrow$ Distance from the origin)

$\therefore$ $l^2 + m^2 + n^2 = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1$

$l^2 + m^2 + n^2 = 1$

(or) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

<u>Note</u> :-

The direction cosines of the x axis are (1,0,0)

The direction cosines of the y axis are (0,1,0)

The direction cosines of the z axis are (0,0,1)

Direction ratios

Any quantities, which are proportional to the direction cosines of a line, are called direction ratios of that line. Direction ratios are denoted by a, b, c.

If l, m, n are direction cosines an a, b, c are direction ratios then

$a \propto l, b \propto m, c \propto n$

(ie) $a = kl, b = km, c = kn$

(ie) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k$ (Constant)

(or) $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{1}{k}$ (Constant)

To find direction cosines if direction ratios are given

If a, b, c are the direction ratios then direction cosines are

$$\left.\begin{aligned} \frac{l}{a} = \frac{1}{k} \Rightarrow \quad & l = \frac{a}{k} \\ \text{similarly} \quad & m = \frac{b}{k} \\ & n = \frac{c}{k} \end{aligned}\right\} \qquad (1)$$

$l^2+m^2+n^2 = \frac{1}{k^2}(a^2+b^2+c^2)$

(ie) $1 = \frac{1}{k^2}(a^2+b^2+c^2)$

$$\Rightarrow k^2 = a^2+b^2+c^2$$

Taking square root on both sides

K = $\sqrt{a^2+b^2+c^2}$

$$\therefore l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

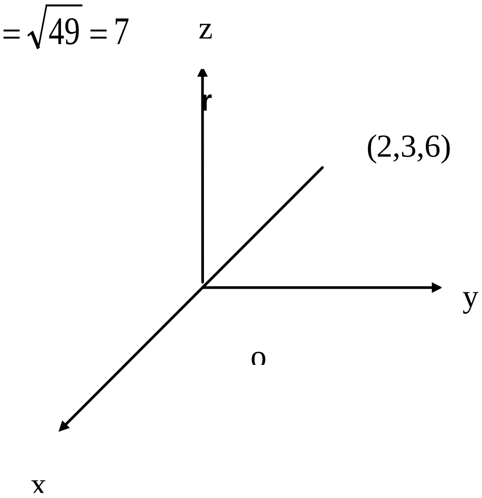
Problem

1. Find the direction cosines of the line joining the point (2,3,6) & the origin.

Solution

By the distance formula

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$



Direction Cosines are

$l = \cos \propto = \dfrac{x}{r} = \dfrac{2}{7}$

$m = \cos \beta = \dfrac{y}{r} = \dfrac{3}{7}$

$n = \cos \gamma = \dfrac{z}{r} = \dfrac{6}{7}$

2. Direction ratios of a line are 3,4,12. Find direction cosines

Solution

Direction ratios are 3,4,12

(ie) a = 3, b = 4, c = 12

Direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{3}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{3}{\sqrt{169}} = \frac{3}{13}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{4}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{4}{\sqrt{169}} = \frac{4}{13}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{12}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{12}{\sqrt{169}} = \frac{12}{13}$$

Note

1) The direction ratios of the line joining the two points A(x_1, y_1, z_1) & B (x_2, y_2, z_2) are ($x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$)

2) The direction cosines of the line joining two points A (x_1, y_1, z_1) & B (x_2, y_2, z_2) are $\dfrac{x_2 - x_1}{r}, \dfrac{y_2 - y_1}{r}, \dfrac{z_2 - z_1}{r}$

r = distance between AB.